Geometric Phase in a Two Energy Level k-Photon Jaynes-Cummings Model with Imaginary Photon Process

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Abstract By using the Lewis-Riesenfeld invariant theory, we have studied the dynamical phase and the geometric phase in a two energy level k-photon Jaynes-Cummings model with imaginary photon process. We find that the geometric phase in a cycle case is independent of the frequency of the photon field, the coupling coefficient between photons and atoms, and the atom transition frequency. We predict the physical effect of the geometric phase in the imaginary photon process may be measured.

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity,

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and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

It is known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is used to treat time-dependent system, and it is generalized by introducing the concept of basic invariants and use to study the geometric phase [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. Based on the studies of the Jaynes-Cummings model [28–33], and by using the Lewis-Riesenfeld invariant theory, we shall study the geometric phase in a two energy level k-photon Jaynes-Cummings model with imaginary photon process.

2 Model

The Hamiltonian of the simplest time-dependent two energy level k-photon Jaynes-Cummings model with imaginary photon process for two-level atoms interacting with light can be written as

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \omega_0(t)\hat{\sigma}_z + g(t)(\hat{a}^k\hat{\sigma}_+ + \hat{a}^{\dagger k}\hat{\sigma}_-) + g(t)(\hat{a}^{\dagger k}\hat{\sigma}_+ e^{2i\omega(t)t} + \hat{a}^k\hat{\sigma}_- e^{-2i\omega(t)t}), \quad (1)$$

where the last term $g(t)(\hat{a}^{\dagger k}\hat{\sigma}_+ e^{2i\omega(t)t} + \hat{a}^k\hat{\sigma}_- e^{-2i\omega(t)t})$ denotes the imaginary photon process, $\hat{a}$ and $\hat{a}^\dagger$ are the photon annihilation and creation operators. $\hat{\sigma}_+(\hat{\sigma}_-)$ are the pseudospin operators for the atom defined as $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$ with $\hat{\sigma}_x$ and $\hat{\sigma}_y$ being the Pauli matrices, $g(t)$ stands for the coupling strength between the atom and the light field, $\omega(t)$ is the frequency of the electromagnetic wave, and $\omega_0(t)$ is the energy difference between two levels of the atom in the unit $\hbar = 1$.

It is known that the imaginary photons are absorbed by the atoms in a rapid speed. If we consider a shorter time scale, the geometric phase in the imaginary photon process may be observed, which is the starting point of this paper. Based on this idea, we consider the following Hamiltonian of the system

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \omega_0(t)\hat{\sigma}_z + g(t)(\hat{a}^{\dagger k}\hat{\sigma}_+ e^{2i\omega(t)t} + \hat{a}^k\hat{\sigma}_- e^{-2i\omega(t)t}). \quad (2)$$

If introducing the supersymmetric generators $\hat{P}_+ = \hat{a}^{\dagger k}\hat{\sigma}_+$ and $\hat{P}_- = \hat{a}^k\hat{\sigma}_-$, one has

$$\{\hat{P}_+, \hat{P}_-\} = \begin{pmatrix} 4\hat{a}^{\dagger k}\hat{a}^k & 0 \\ 0 & 4\hat{a}^k\hat{a}^{\dagger k} \end{pmatrix} \equiv \hat{\tilde{M}}. \quad (3)$$

It is easy to find that $\hat{\tilde{M}}$, $\hat{a}^\dagger\hat{a}$, $\hat{\sigma}_\pm$, $\hat{\sigma}_z$, and $\hat{P}_\pm$, are the supersymmetric generators and they construct the supersymmetric Lie algebra, namely

$$\hat{P}_-^2 = \hat{P}_+^2 = 0, \quad [\hat{P}_+, \hat{P}_-] = \hat{\tilde{M}}\hat{\sigma}_z, \quad [\hat{\tilde{M}}, \hat{P}_-] = [\hat{\tilde{M}}, \hat{P}_+] = [\hat{\tilde{M}}, \hat{\sigma}_z] = 0, \quad (4)$$

$$[\hat{\tilde{M}}, \hat{a}^\dagger\hat{a}] = 0, \quad \{\hat{P}_-, \hat{\sigma}_z\} = \{\hat{P}_+, \hat{\sigma}_z\} = 0, \quad [\hat{\sigma}_z, \hat{a}^\dagger\hat{a}] = 0, \quad (5)$$

$$\hat{\sigma}_z(\hat{P}_+ - \hat{P}_-) = (\hat{P}_+ + \hat{P}_-), \quad (\hat{P}_+ - \hat{P}_-)^2 = -\hat{\tilde{M}}, \quad (6)$$

$$[\hat{P}_-, \hat{a}^\dagger\hat{a}] = k\hat{P}_-, \quad [\hat{P}_+, \hat{a}^\dagger\hat{a}] = -k\hat{P}_+, \quad [\hat{P}_-, \hat{\sigma}_z] = 2\hat{P}_-, \quad [\hat{P}_+, \hat{\sigma}_z] = -2\hat{P}_+, \quad (7)$$

where $\{,\}$ stands for the anticommuting bracket. Equation (2) becomes

$$\hat{H} = \omega(t)\hat{a}^{\dagger}\hat{a} + \omega_0(t)\hat{\sigma}_z + g(t)(\hat{P}_+ e^{2i\omega(t)t} + \hat{P}_- e^{-2i\omega(t)t}). \tag{8}$$

It is easy to find that

$$\hat{\tilde{M}} \begin{pmatrix} |n+k\rangle \\ |n\rangle \end{pmatrix} = 4\frac{(n+k)!}{n!} \begin{pmatrix} |n+k\rangle \\ |n\rangle \end{pmatrix}, \tag{9}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra constructed by operators $\hat{\tilde{M}}, \hat{a}^{\dagger}\hat{a}, \hat{P}_{\pm}, \hat{\sigma}_{\pm}$, and $\hat{\sigma}_z$. Below, we will replace operator $\hat{\tilde{M}}$ with the particular eigenvalue $4\frac{(n+k)!}{n!}$.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{10}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given by

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{11}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial |\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{12}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (12) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (12). Then the general solution of the Schrödinger equation (12) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{14}$$

where

$$\delta_n(t) = \int_0^t dt' \left\langle \lambda_n, t' \left| i\frac{\partial}{\partial t'} - \hat{H}(t') \right| \lambda_n, t' \right\rangle, \tag{15}$$

and $C_n = \langle \lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (8), we can define the following invariant

$$\hat{I}(t) = \alpha(t)\hat{P}_+ + \alpha^*(t)\hat{P}_- + \beta(t)\hat{\sigma}_z. \tag{16}$$

Substituting (8) and (16) into (10), one has the auxiliary equations

$$i\dot{\alpha}(t) - [2\omega_0(t) + k\omega(t)]\alpha(t) + 2\beta(t)g(t)e^{2i\omega(t)t} = 0, \tag{17}$$

$$i\dot{\beta}(t) + g(t)\bar{M}[\alpha(t)e^{-2i\omega(t)t} - \alpha^*(t)e^{2i\omega(t)t}] = 0, \tag{18}$$

where dot denotes the time derivative, and $\bar{M}$ denotes the eigenvalue of operator $\hat{\bar{M}}$.

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t) = \exp[\xi(t)\hat{P}_- - \xi^*(t)\hat{P}_+]$. It is easy to find that when there exist the following relations

$$\sin(2\sqrt{\bar{M}}|\xi(t)|) = \frac{\sqrt{\bar{M}}[\alpha(t)\xi(t) + \alpha^*(t)\xi^*(t)]}{2|\xi(t)|}, \qquad \beta(t) = \cos(2\sqrt{\bar{M}}|\xi(t)|), \tag{19}$$

and

$$\alpha(t) - \frac{\beta(t)\xi^*(t)}{\sqrt{\bar{M}}|\xi(t)|}\sin(2\sqrt{\bar{M}}|\xi(t)|) + \frac{[\alpha(t)|\xi(t)|^2 + \alpha^*(t)\xi^{*2}(t)]}{2|\xi(t)|^2}[\cos(2\sqrt{\bar{M}}|\xi(t)|) - 1] = 0, \tag{20}$$

a time-independent invariant appears

$$\hat{I}_V \equiv \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t) = \hat{\sigma}_z. \tag{21}$$

According to (19) and (20), we can select

$$\xi(t) = \frac{\theta(t)}{\sqrt{\bar{M}}}\exp[-i\gamma(t)], \qquad \alpha(t) = \frac{\sin\theta(t)}{\sqrt{\bar{M}}}\exp[i\gamma(t)], \quad \theta(t) = 2\sqrt{\bar{M}}|\xi(t)|. \tag{22}$$

From (22), the invariant $\hat{I}(t)$ in (16) becomes

$$\hat{I}(t) = \frac{\sin\theta(t)}{\sqrt{\bar{M}}}\{\exp[i\gamma(t)]\hat{P}_+ + \exp[-i\gamma(t)]\hat{P}_-\} + \cos\theta(t)\hat{\sigma}_z. \tag{23}$$

By using the Baker-Campbell-Hausdorff formula [34]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t} = \frac{\partial\hat{L}}{\partial t} + \frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right] + \frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right] + \frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right], \hat{L}\right] + \cdots, \tag{24}$$

with $\hat{V}(t) = \exp[\hat{L}(t)]$, it is easy to find that when the following equation holds

$$i\dot{\xi}(t) + i\frac{\xi(t)[\dot{\xi}(t)\xi^*(t) - \dot{\xi}^*(t)\xi(t)]}{4\sqrt{\bar{M}}|\xi(t)|^3}[\sin(2\sqrt{\bar{M}}|\xi(t)|) - 2\sqrt{\bar{M}}|\xi(t)|]$$

$$+ \frac{[2\omega_0(t) + \omega(t)]\xi(t)}{2\sqrt{\bar{M}}|\xi(t)|}\sin(2\sqrt{\bar{M}}|\xi(t)|) - \frac{ge^{-2i\omega(t)t}}{2}[\cos(2\sqrt{\bar{M}}|\xi(t)|) - 1]$$

$$- ge^{-2i\omega(t)t} - \frac{g\xi^2(t)e^{2i\omega(t)t}}{2|\xi(t)|^2}[\cos(2\sqrt{\bar{M}}|\xi(t)|) - 1] = 0, \tag{25}$$

one has

$$\hat{H}_V(t) = \hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t) - i\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t}$$

$$= \omega(t)\hat{a}^\dagger\hat{a} + \left\{ \omega_0(t)\cos\theta(t) - \frac{\omega(t)}{2}[1 - \cos\theta(t)] \right.$$

$$\left. + \sqrt{\bar{M}}g(t)\sin\theta(t)\cos[2\omega(t)t - \gamma(t)] \right\}\hat{\sigma}_z - \frac{\dot{\gamma}(t)}{2}[1 - \cos\theta(t)]\hat{\sigma}_z. \quad (26)$$

The eigenstates of operator $\hat{\sigma}_z$ corresponding to the eigenvalues $\sigma_z = +1$, $\sigma_z = -1$ are $\binom{1}{0}$ and $\binom{0}{1}$, respectively. The eigenstate of operator $\hat{\bar{M}}$ is $\binom{|n+k\rangle}{|n\rangle}$. So we can obtain two particular solutions of the time-dependent Schrödinger equation (12), respectively. For $\sigma_z = +1$, one has

$$|\psi_{\sigma_z=+1}(t)\rangle = \exp\left\{ -i\int_0^t [\dot{\delta}^d_{\sigma_z=+1}(t') + \dot{\delta}^g_{\sigma_z=+1}(t')]dt' \right\}\hat{V}(t)\binom{|n+k\rangle}{0}, \quad (27)$$

where

$$\dot{\delta}^d_{\sigma_z=+1}(t') = (n+k)\omega(t') + \omega_0(t')\cos\theta(t') - \frac{\omega(t')}{2}[1 - \cos\theta(t')]$$

$$+ 2\sqrt{\frac{(n+k)!}{n!}}g(t')\sin\theta(t')\cos[2\omega(t')t' - \gamma(t')], \quad (28)$$

$$\dot{\delta}^g_{\sigma_z=+1}(t') = -\frac{\dot{\gamma}(t')}{2}[1 - \cos\theta(t')], \quad (29)$$

for $\sigma_z = -1$,

$$|\psi_{\sigma_z=-1}(t)\rangle = \exp\left\{ -i\int_0^t [\dot{\delta}^d_{\sigma_z=-1}(t') + \dot{\delta}^g_{\sigma_z=-1}(t')]dt' \right\}\hat{V}(t)\binom{0}{|n\rangle}, \quad (30)$$

where

$$\dot{\delta}^d_{\sigma_z=-1}(t') = n\omega(t') - \omega_0(t')\cos\theta(t') + \frac{\omega(t')}{2}[1 - \cos\theta(t')]$$

$$- 2\sqrt{\frac{(n+k)!}{n!}}g(t')\sin\theta(t')\cos[2\omega(t')t' - \gamma(t')], \quad (31)$$

$$\dot{\delta}^g_{\sigma_z=-1}(t') = \frac{\dot{\gamma}(t')}{2}[1 - \cos\theta(t')]. \quad (32)$$

From (28), (29), (31) and (32), we conclude that the dynamical and the geometric phase factors of the system are $\exp[-i\int_0^t \dot{\delta}^d_{\sigma_z}(t')dt']$ and $\exp[-i\int_0^t \dot{\delta}^g_{\sigma_z}(t')dt']$ with $\sigma_z = \pm 1$, respectively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{I}(t)$ and let $\theta(t) = $ constant, one has from (29) and (32)

$$\delta^g_{\sigma_z}(T) = -\frac{\sigma_z}{2}2\pi(1 - \cos\theta) \quad (\sigma_z = \pm 1), \quad (33)$$

here $2\pi(1 - \cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{I}(t)$. It is pointed out that the geometric phase in the cycle case is independent of the frequency $\omega(t)$ of the photon field, the coupling coefficient $g(t)$ between photons and atoms, and the atom transition frequency $\omega_0(t)$.

4 Conclusions

In this letter, we have studied the dynamical phase and the geometric phase in a two energy level k-photon Jaynes-Cummings model with imaginary photon process. We find that the geometric phase in a cycle case is independent of the frequency $\omega(t)$ of the photon field, the coupling coefficient $g(t)$ between photons and atoms, and the atom transition frequency $\omega_0(t)$. We predict the physical effect of the geometric phase in the imaginary photon process may be measured.

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